JOURNAL OF APPROXIMATION THEORY 22, 331–339 (1978)

Integral Equations and the Interior Dirichlet Potential Problem

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Communicated by Oved Shisha

Received February 6, 1976

The two standard approaches for reformulating the interior Dirichlet potential problem as a boundary integral equation of the second kind are discussed. The integral equation derived from the representation of the solution as a double layer is shown to be more general than the one derived from Green's theorem. The boundary integral equation of the latter method, however, has definite analytical and numerical value. From it a new integral equation is derived whose solution can be represented as a convergent Neumann series and it is shown that the Green's function of the first kind can be obtained from it. An example is supplied to illustrate the method.

1. INTRODUCTION

The method of integral equations has long been recognized both for its numerical and theoretical importance in treating boundary value problems. One such problem receiving considerable attention in mathematical physics is the three-dimensional interior Dirichlet problem for Laplace's equation. For those geometries where Laplace's equation separates, the solution can be expanded in terms of eigenfunctions. For arbitrary surfaces, however, other techniques must be employed. While finite difference methods are practical for solving two-dimensional problems, they have limited use in the case of three-dimensional regions. One advantage of integral equations, is that it reduces the problem from the entire domain of interest to one involving only its boundary.

The interior Dirichlet problem may be reformulated as a Fredholm integral equation of either the first or second kind (e.g., see Noble [16] or Kleinman and Roach [8]). The integral equation of the first kind does not lead to analytical results, and numerically, this type of equation leads to certain difficulties not encountered with one of the second kind. This stems from the fact that the range of a compact, nondegenerate operator, is always a non-closed subspace of some Hilbert space. There are, however, approximation and regularization methods which deal with these difficulties. Nashed and Wahba [15] have shown that the range of a compact operator can be viewed as a closed subspace with respect to a new inner product, even though it is

0021-9045/78/0224-0331$02.00/0

a nonclosed subspace of L^2. Approximation methods can then be developed based on this observation (see [15]). Howland [5] uses a method of regularization to reformulate the first kind equation, obtained from representing the solution as a single-layer potential, as one of the second kind, by finding an appropriate reducing operator and establishing some important operator identities. For other regularization methods for integral equations of the first kind see [13, 14]. There are, however, two approaches for reformulating the Dirichlet problem as an integral equation of the second kind, which have received considerable attention. One method is founded on the assumption that the solution may be represented as a double-layer distribution with an unknown density function (e.g. see [17–19]). The other approach is based on Green's theorem, where the normal derivative is taken as the field point approaches the boundary (see [16] and [8]). The resulting integral equations of the two methods are adjoint in the L^2 sense where the kernel function in one is obtained by interchanging the two variables in the other one. In view of this, little attention has been given to the differences between the two methods.

It is well known (e.g., see [10, Chap. 12]) that the interior Dirichlet problem has a unique solution for smooth, simply connected, compact surfaces, when the function given on the boundary is continuous. It will be shown in the next section, that the Green's theorem method is based on the existence of a certain normal derivative of a double layer, where the density function is the prescribed function. For this derivative to exist and be in the range of the integral operator, however, the given function must have more smoothness than just continutiy. This is in contrast to the layer approach, where just continuity is needed. It is felt that this distinction between the two methods is important and at least from an operator point of view, the layer approach is preferable to the Green's theorem approach.

Despite this limitation, however, the integral equation derived from Green's theorem has definite analytical and numerical value. When the prescribed function is smooth enough, it is shown in Section 3, that the normal derivative of the solution on the surface may be expressed as a Neumann series and hence the solution in the region of interest may be found. Furthermore, it is indicated that this method can always be used to calculate the Green's function of the first kind. In the last section, the method is illustrated by constructing the Green's function for a sphere and it is demonstrated that the classical solution is obtained.

2. Basic Equations

Consider a region V_i in E^3 bounded by a compact Lyapunoff surface S with outward normal $\hat{n}$. Let V_e represent the region exterior to S. With

respect to a rectangular coordinate system, with origin $\mathbf{0} \in V_i$, a point (x_1, x_2, x_3) will be denoted by $\mathbf{x}$. The distance between two points $\mathbf{x}$ and $\mathbf{y}$ will be denoted by $r(\mathbf{x}, \mathbf{y})$ or simply r.

The problem we are concerned with is to find the scalar function $u(\mathbf{x})$ which satisfies Laplace's equation in V_i and is prescribed on S, i.e.,

$$\begin{aligned} \nabla^2 u(\mathbf{x}) &= 0, \qquad \mathbf{x} \in V_i, \\ u(\mathbf{x}) &= f(\mathbf{x}), \qquad \mathbf{x} \in S, \end{aligned} \tag{2.1}$$

where $f(\mathbf{x})$ is a given continuous function defined on S. For convenience we shall hereafter refer to this problem as P.

P may be reformulated as a Fredholm integral equation of the second kind either by representing $u(\mathbf{x})$ as a double-layer distribution and taking the limit as $\mathbf{x}$ approaches the boundary from inside or by using Green's theorem. In the layer approach, it is assumed that

$$u(\mathbf{x}) = \frac{1}{2\pi} \int_S \mu(\mathbf{y}) \frac{\partial}{\partial n_\mathbf{y}} \frac{1}{r} \, dS_\mathbf{y}, \qquad \mathbf{x} \in V_i. \tag{2.2}$$

In taking the limit as $\mathbf{x}$ tends to S and using the well-known jump discontinuity condition of a double layer, one obtains (see [19, p. 388])

$$f(\mathbf{x}) = -\mu(\mathbf{x}) + \frac{1}{2\pi} \int_S \mu(\mathbf{y}) \frac{\partial}{\partial n_\mathbf{y}} \frac{1}{r} \, dS_\mathbf{y} \tag{2.3}$$

where the boundary condition of P has been used. $\mu(\mathbf{x})$ is the density of the moment of the double layer (see [19, p. 382]). This is the usual integral reformulation to P found in most texts which treat this topic (e.g., see [3, p. 179; 7, p. 286; and 17, p. 617]).

From Green's identities (see [1, Vol. II, p. 256]) we have

$$\begin{aligned} -\frac{1}{4\pi} \int_S \left\{ u(\mathbf{y}) \frac{\partial}{\partial n_\mathbf{y}} \frac{1}{r} - \frac{1}{r} \frac{\partial u}{\partial n} \right\} dS_\mathbf{y} &= 0, \qquad \mathbf{x} \in V_e \\ &= \tfrac{1}{2} u(\mathbf{x}), \qquad \mathbf{x} \in S \\ &= u(\mathbf{x}), \qquad \mathbf{x} \in V_i \end{aligned} \tag{2.4}$$

and it follows that once $\partial u/\partial n$ is known on S, $u(\mathbf{x})$ is known everywhere in V_i. Substituting the boundary condition into this representation, and taking the normal derivative as $\mathbf{x}$ approaches the surface S from points in V_i and then using the jump-discontinuity condition for taking the normal derivative of a single-layer potential, the following result is obtained:

$$\frac{\partial u}{\partial n}(\mathbf{x}) = F(\mathbf{x}) + \frac{1}{2} \frac{\partial u}{\partial n}(\mathbf{x}) + \frac{1}{4\pi} \int_S \frac{\partial u}{\partial n}(\mathbf{y}) \frac{\partial}{\partial n_\mathbf{x}} \frac{1}{r} \, dS_\mathbf{y} \tag{2.5}$$

which immediately leads to a Fredholm integral equation of the second kind,

$$\frac{\partial u}{\partial n}(\mathbf{x}) = 2F(\mathbf{x}) + \frac{1}{2\pi}\int_S \frac{\partial u}{\partial n}(\mathbf{y})\,\frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{r}\,dS_{\mathbf{y}} \tag{2.6}$$

where

$$F(\mathbf{x}) = \lim_{x \to S^-}\left\{-\frac{1}{4\pi}\frac{\partial}{\partial n_{\mathbf{x}}}\int_S f(\mathbf{y})\,\frac{\partial}{\partial n_{\mathbf{y}}}\frac{1}{r}\,dS_{\mathbf{y}}\right\}. \tag{2.7}$$

Since the Green's theorem approach makes no a priori assumption about what form the solution must have, one might conclude that this method is preferable to the layer representation. Because each of the kernel functions in the integral operators in (2.3) and (2.6) is obtained by interchanging the variables in the other kernel function, it follows that each integral is just the adjoint operator of the other. Hence it might be concluded that these methods are equivalent in the sense that they can both be used to solve P.

This, however, is not the case. In the Green's theorem method it is essential that the function $F(\mathbf{x})$ given in (2.7) be defined and moreover lie in the range of $I - K^*$ where I is the identity map and

$$\left(K^*\frac{\partial u}{\partial n}\right)(\mathbf{x}) = \frac{1}{2\pi}\int_S \frac{\partial u}{\partial n}(\mathbf{y})\,\frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{r}\,dS_{\mathbf{y}}\,. \tag{2.8}$$

A condition sufficient to guarantee the existence of the normal derivative of a double layer on a closed Lyapunoff surface is that the density function be differentiable on S (see [3, p. 73]). Furthermore, it can be shown (see [3, p. 71]) that just continuity or even Hölder continuity of the density function is not sufficient for the existence of the normal derivative. For our purposes $f(\mathbf{x})$ must be smooth enough so that $F(\mathbf{x})$ is continuous for $\mathbf{x} \in S$. Maue [9] and later Mitzner [11] derived an alternative representation for the normal derivative of a double layer

$$\frac{\partial}{\partial n_{\mathbf{x}}}\int_S f(\mathbf{y})\,\frac{\partial}{\partial n_{\mathbf{y}}}\frac{1}{r}\,dS_{\mathbf{y}} = \int_S (\hat{n}_{\mathbf{y}} \times \nabla f(\mathbf{y})) \cdot \left(\hat{n}_{\mathbf{x}} \times \nabla_{\mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{y}|}\right) dS_{\mathbf{y}}\,. \tag{2.9}$$

If each first partial derivative of f is continuous, then each partial derivative of $1/|\mathbf{x} - \mathbf{y}|$ has as coefficient a continuous function and it follows that (see [3, Chap II, Sect. 8, especially Eq. (38)]) the boundary integral on the right-hand side of (2.9) is continuous. It should also be noted that the normal derivative of a double layer is continuous, for f sufficiently smooth, as the field point $\mathbf{x}$ approaches the boundary from both the exterior and interior regions, provided f is sufficiently smooth. This continuous behavior is in contrast to that of the double layer, which, even for density functions possessing continuous derivatives, is discontinuous on S as $\mathbf{x}$ approaches the surface from both V_i and V_e.

In view of the fact that the prescribed function $f(\mathbf{x})$ must have more smoothness when the Green's theorem method is used than when the double-layer distribution approach is employed, it is felt that the latter method is preferable. This conclusion is even more noteworthy, in view of the fact that an opposite result is arrived at by Kleinman and Roach [8] for exterior scattering problems. Their conclusion is reached, however, to circumvent the appearance of interior eigenvalues in the integral equations for exterior problems and the question of the smoothness of the boundary conditions is avoided. Despite this limitation on the Green's theorem approach, the integral equation in (2.6) does have definite analytical value which is demonstrated in the next section. One important function having continuous first partial derivatives is

$$f(\mathbf{x}) = -\frac{1}{|\mathbf{x} - \mathbf{x}_0|}, \qquad \mathbf{x}_0 \in V_i, \quad \mathbf{x} \in S$$

and the solution to P in this case permits one to obtain the Green's function of the first kind to Laplace's equation. In what follows, it will be assumed that $f(\mathbf{x})$ has continuous partial derivatives on S.

3. A Neumann Series

In this section it will be shown that the integral equation in (2.5) can be solved iteratively. Corresponding to (2.6) we consider the integral equation

$$\frac{\partial u}{\partial n}(\mathbf{x}) = 2F(\mathbf{x}) + \lambda \int_S \frac{\partial u}{\partial n}(\mathbf{y}) \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{2\pi r} \, dS_{\mathbf{y}} . \tag{3.1}$$

It can be shown (see [7, pp. 309–312]) that the eigenvalues λ_i of the corresponding homogeneous equation to (3.1) are never less than 1 in absolute value, and that $\lambda = 1$ is a regular value, while $\lambda = -1$ is an eigenvalue. Thus (3.1) is invertible for $\lambda = 1$, however, for $f(\mathbf{x})$ arbitrary, it is not possible in general to represent the solution as a convergent Neumann series. We now employ the method of shifting eigenvalues (see [6, p. 118]) to obtain an integral equation which can be solved iteratively. Let

$$\lambda = \frac{\eta}{1 - \eta} . \tag{3.2}$$

With K^* defined in (2.8), 3.1) becomes

$$(1 - \eta) \frac{\partial u}{\partial n}(\mathbf{x}) = 2(1 - \eta) F(\mathbf{x}) + \eta \left(K^* \frac{\partial u}{\partial n} \right)(\mathbf{x}) \tag{3.3}$$

or

$$\frac{\partial u}{\partial n}(\mathbf{x}) = 2(1 - \eta) F(\mathbf{x}) + \eta \left\{ \frac{\partial u}{\partial n}(\mathbf{x}) + \left(K^* \frac{\partial u}{\partial n} \right)(\mathbf{x}) \right\}. \tag{3.4}$$

From (3.2) we have that $\eta = \lambda/(1 + \lambda)$ and it is seen that (3.4) has a unique solution for values of η such that

$$| \eta | < \inf_{i=1,2,\ldots} \frac{| \lambda_i |}{| 1 + \lambda_i |} = | \eta_1 | \tag{3.5}$$

where η_1 is the eigenvalue, having smallest absolute value, of the homogeneous equation corresponding to (3.4). Since $\lambda_i \neq 1$, it follows that

$$\tfrac{1}{2} < | \eta_1 | \tag{3.6}$$

and hence for $\eta = \frac{1}{2}$ the integral equation (3.4) may be solved iteratively. For this choice of η, however, the integral equation in (3.4) is identical to the one in (2.5) and consequently the solution may be expressed as the Neumann series

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \sum_{j=0}^{\infty} \left(\frac{1}{2} \right)^j (I + K^*)^j F(\mathbf{x}). \tag{3.7}$$

A similar result was obtained by Neumann (see [2, p. 201]) for solving the interior Dirichlet problem for convex contours in E^2, based on the assumption that the solution could be represented as a double-layer distribution. A generalization of Neumann's method to arbitrarily shaped contours is given in [4, p. 135] where the method of shifting eigenvalues is used. The result there can also be extended to E^3. The integral operator appearing in the integral equation in Neumann's method is the adjoint of the one in (2.5). Thus while the method here is analogous to Neumann's method, the convergence of the series (3.7) is apparently a new result.

4. An Example

In this section the Green's function of the first kind for a sphere of radius a is obtained via the Neumann series (3.7). Thus we let

$$f(\mathbf{x}) = -\frac{1}{| \mathbf{x} - \mathbf{x}_0 |}, \qquad \mathbf{x}_0 \in V_i \tag{4.1}$$

and the Green's function is just

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{| \mathbf{x} - \mathbf{x}_0 |} + u(\mathbf{x}, \mathbf{x}_0), \qquad x, x_0 \in V_i$$

where $u(\mathbf{x}, \mathbf{x}_0)$ is found from the integral representation (2.4).

To evaluate the series (3.7) we must first find $F(\mathbf{x})$, which may be determined from (2.7) or from its alternative integral representation in (2.9). From [18, p. 85] we have

$$\frac{1}{|x - x_0|} = \sum_{n=0}^{\infty} \frac{{r_0}^n}{r^{n+1}} P_n(\cos\alpha) \tag{4.2}$$

for $r_0 \equiv r(\mathbf{0}, \mathbf{x}) < r(\mathbf{0}, \mathbf{x}) \equiv r$, where $\cos\alpha = \hat{r}(\mathbf{0}, \mathbf{x}) \cdot \hat{r}(\mathbf{0}, \mathbf{x}_0)$ and where $\hat{r}(\mathbf{0}, \mathbf{x})$ denotes a unit vector from $\mathbf{0}$ to $\mathbf{x}$. For $\mathbf{x} \in V_i$ and the fact that on the sphere $\partial/\partial n_{\mathbf{y}} = \partial/\partial a$, we have

$$\begin{aligned}\int_S \frac{1}{|\mathbf{y} - \mathbf{x}_0|} \frac{\partial}{\partial n_{\mathbf{y}}} \frac{1}{|\mathbf{y} - \mathbf{x}|} dS_{\mathbf{y}} \\ = \int_S \left\{\sum_{n=0}^{\infty} \frac{{r_0}^n}{a^{n+1}} P_n(\cos\beta)\right\}\left\{-\sum_{m=0}^{\infty} \frac{(m+1)\, r^m}{a^{m+2}} P_m(\cos\gamma)\right\} dS_{\mathbf{y}}\end{aligned} \tag{4.3}$$

where $\cos\beta = \hat{r}(\mathbf{0}, \mathbf{y}) \cdot \hat{r}(\mathbf{0}, \mathbf{x}_0)$ and $\cos\gamma = \hat{r}(\mathbf{0}, \mathbf{y}) \cdot \hat{r}(\mathbf{0}, \mathbf{x})$. Using the orthogonality of the Legendre polynomials it follows that

$$\int_S \frac{1}{|\mathbf{y} - \mathbf{x}_0|} \frac{\partial}{\partial n_{\mathbf{y}}} \frac{1}{|\mathbf{y} - \mathbf{x}|} dS_{\mathbf{y}} = -4\pi \sum_{n=0}^{\infty} \frac{{r_0}^n r^n}{a^{2n+1}} P_n(\cos\alpha). \tag{4.4}$$

Taking the normal derivative of (4.4) as $\mathbf{x}$ approaches S from V_i and using the fact that the normal derivative of a double layer is continuous as the field point approaches the boundary, the next result is obtained,

$$\begin{aligned}F(\mathbf{x}) &= -\frac{1}{4\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \int_S \frac{1}{|\mathbf{y} - \mathbf{x}_0|} \frac{\partial}{\partial n_{\mathbf{y}}} \frac{1}{|\mathbf{y} - \mathbf{x}|} dS_{\mathbf{y}} \\ &= \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \frac{{r_0}^n}{a^{n+2}} P_n(\cos\alpha).\end{aligned} \tag{4.5}$$

Next we determine $\frac{1}{2}(I + K^*) F(\mathbf{x})$. Since $|\mathbf{x}| = |\mathbf{y}| = a$, we employ an average of two expansions of $1/|\mathbf{x} - \mathbf{y}|$, similar to (4.2), with $|\mathbf{y}| = a_+ > |\mathbf{x}|$ in one and $|\mathbf{y}| = a_- < |\mathbf{x}|$ in the other. Thus

$$\begin{aligned}\frac{1}{2}(I + K^*) F(\mathbf{x}) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \frac{{r_0}^n}{a^{n+2}} P_n(\cos\alpha) \\ + \frac{1}{4\pi} \int_S \left\{\sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \frac{{r_0}^n}{a^{n+2}} P_n(\cos\beta)\right\} \\ \times \left\{\frac{1}{2} \sum_{m=0}^{\infty} \frac{m a^{m-1}}{a_+^{m+1}} P_m(\cos\gamma) \right. \\ \left. + \frac{1}{2} \sum_{m=0}^{\infty} \frac{-(m+1)\, {a_-}^m}{a^{m+2}} P_m(\cos\gamma)\right\} dS.\end{aligned} \tag{4.6}$$

From the orthogonality of the Legendre polynomials and subsequently letting $a_+ = a_- = a$, it follows that

$$\frac{1}{2}(I + K^*)F(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{n(n+1)}{2n+1} \cdot \frac{n}{2n+1} \frac{r_0^n}{a^{n+2}} P_n(\cos\alpha). \tag{4.7}$$

By an induction argument, it can be shown that

$$\left(\frac{1}{2}\right)^j (I + K^*)^j F(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \left(\frac{n}{2n+1}\right)^j \frac{r_0^n}{a^{n+2}} P_n(\cos\alpha). \tag{4.8}$$

Summing the geometric series with terms $n/2n + 1$, it follows that

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \sum_{n=0}^{\infty} n \frac{r_0^n}{a^{n+1}} P_n(\cos\alpha), \qquad \mathbf{x} \in S. \tag{4.9}$$

To find $u(\mathbf{x})$, $\mathbf{x} \in V_i$, we must substitute the series (4.9) into the integral representation (2.4) for $\partial u/\partial n$ and let $u = -1/|\mathbf{y} - \mathbf{x}_0|$ on S.

Thus

$$u(\mathbf{x}) = -\frac{1}{4\pi} \int_S \left\{ -\sum_{n=0}^{\infty} \frac{r_0^n}{a^{n+1}} P_n(\cos\beta) \right\} \left\{ -\sum_{m=0}^{\infty} \frac{(m+1)r^m}{a^{m+2}} P_m(\cos\gamma) \right\} dS_{\mathbf{y}}$$

$$+ \int_S \left\{ \sum_{n=0}^{\infty} \frac{nr_0^n}{a^{n+2}} P_n(\cos\beta) \right\} \left\{ \sum_{m=0}^{\infty} \frac{r^m}{a^{m+1}} P_m(\cos\gamma) \right\} dS_{\mathbf{y}} \tag{4.10}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{r_0^n r^n}{a^{2n+1}} P_n(\cos\alpha). \tag{4.11}$$

The series representation for the Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = \sum_{n=0}^{\infty} \frac{r_-^n}{r_+^n} P_n(\cos\alpha) - \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{r^n r_0^n}{a^{2n+1}} P_n(\cos\alpha) \tag{4.12}$$

where $r_- = \min\{r, r_0\}$ and $r_+ = \max\{r, r_0\}$ which corresponds to the result using separation of variables (see, [12, Vol. II, p. 1275]).

REFERENCES

1. R. COURANT AND D. HILBERT, "Methods of Mathematical Physics," Wiley, New York, 1962.
2. E. GOURSAT, "A Course in Mathematical Analysis," Vol. III, Part 1, Dover, New York, 1964.
3. N. M. GUNTER, "Potential Theory and Its Applications to Basic Problems of Mathematical Physics," Frederick Ungar, New York, 1967.

4. W. HAACK AND W. WENDLAND, "Lectures on Partial and Pfaffian Differential Equations," Pergamon Press, New York, 1972.
5. J. L. HOWLAND, Symmetrizing kernels and the equations of the first kind of classical potential theory, *Proc. Amer. Math. Soc.* **19** (1968), 1–7.
6. L. V. KANTOROVICH AND V. I. KRYLOV, "Approximate Methods of Higher Analysis," Interscience, New York, 1958.
7. O. D. KELLOGG, "Foundations of Potential Theory," Dover, New York, 1953.
8. R. E. KLEINMAN AND G. F. ROACH, Boundary integral equations for the three-dimensional Helmholtz equation, *SIAM Rev.* **16** (1974), 214–236.
9. A. W. MAUE, Zur Formulierung Eines Allgemein Beugungsproblems durch eine Integralgleichung, *Z. Phys.* **126** (1949), 601–618.
10. S. G. MIKHLIN, "Mathematical Physics, An Advanced Course", American Elsevier New York, 1970.
11. K. M. MITZNER, Acoustic scattering from an interface between media of greatly different density, *J. Math. Phys.* 7 (1966), 2053–2060.
12. P. M. MORSE AND H. FESHBACH, "Methods of Theoretical Physics," McGraw-Hill, New York, 1953.
13. M. Z. NASHED, Approximate regularized solutions to improperly posed linear integral and operator equations, *in* "Constructive and Computational Methods for Differential and Integral Equations" (D. L. Colton and R. P. Gilbert, Eds.), pp. 289–322, Springer–Verlag Lecture Notes in Mathematics, Vol. 430, 1974.
14. M. Z. NASHED, Aspects of Generalized Inverses in Analysis and Regularization, in *Generalized Inverses and Applications*, Academic Press, New York, 1976, pp. 325–396.
15. M. Z. NASHED AND G. WAHBA, Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind, Math. Comp. **28** (1974), 69–80.
16. B. NOBLE, "Integral Equation Perturbation Methods in Low-Frequency Diffraction, Electromagnetic Waves" (R. E. Langer, Ed.), Univ. of Wisconsin, Madison, 1962.
17. V. I. SMIRNOV, "A Course of Higher Mathematics," Vol. 4, Addison–Wesley, Reading, Mass., 1964.
18. W. J. STERNBERG AND T. L. SMITH, "The Theory of Potential and Spherical Harmonics," Univ. of Toronto Press, Toronto, 1946.
19. A. N. TIKHONOV AND A. A. SAMARSKII, "Equations of Mathematical Physics," Pergamon, New York, 1963.